Infinite Challenge

AP® CALCULUS BC PLUS

Richard N.S. Seong

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- ✓ 1400 carefully crafted questions with full solutions!
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The **College Board** provides free-response questions on past exams with scoring guidelines (https://apcentral.collegeboard.org/courses/ap-calculus-bc/exam/past-exam-questions).
Students are strongly recommended to study the free-response questions in their preparation for the AP exam.

1. AP Calculus AB Review

★ Limits

◆ Existence of Limit

The limit of a function $f(x)$ exists at $x=a$ if and only if the left-hand limit and the right-hand limit of $f(x)$ have the same value of L.

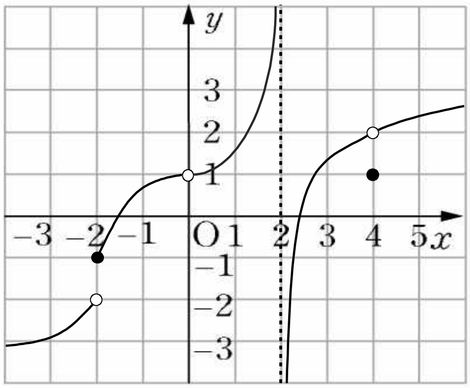
$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

$\lim_{x\to a} f(x) =$ **Does Not Exist (DNE)** if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

$\lim_{x\to a^-} f(x) =$ Left-hand limit

$\lim_{x\to a^+} f(x) =$ Right-hand limit

(Example 1) Use the graph of f to evaluate the following.



① (a) $\lim_{x\to -2^-} f(x) = -2$ (b) $\lim_{x\to -2^+} f(x) = -1$ (c) $\lim_{x\to -2} f(x) =$ DNE (d) $f(-2) = -1$

② (a) $\lim_{x\to 0^-} f(x) = 1$ (b) $\lim_{x\to 0^+} f(x) = 1$ (c) $\lim_{x\to 0} f(x) = 1$ (d) $f(0) =$ DNE

③ (a) $\lim_{x\to 2^-} f(x) = +\infty$ (b) $\lim_{x\to 2^+} f(x) = -\infty$ (c) $\lim_{x\to 2} f(x) =$ DNE (d) $f(2) =$ DNE

④ (a) $\lim_{x\to 4^-} f(x) = 2$ (b) $\lim_{x\to 4^+} f(x) = 2$ (c) $\lim_{x\to 4} f(x) = 2$ (d) $f(4) = 1$

◆ Limit of Indeterminate form: $\frac{0}{0}$

- **Rational Expressions**

Step 1. Factor the numerator and denominator.

Step 2. Cancel the common factor.

(Example 2) Evaluate the following limit.

$$\lim_{x\to 3}\frac{x^2-9}{x^3-27} = \lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x^2+3x+9)}$$

$$= \lim_{x\to 3}\frac{x+3}{x^2+3x+9}$$

$$= \frac{3+3}{3^2+3(3)+9} = \frac{6}{27} = \frac{2}{9}$$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

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Chapter Review Exercises

Do not use a calculator.

1. Evaluate the following limits.

① $\lim_{x\to 1}\frac{(x+1)^3-8}{x-1}$

② $\lim_{x\to 3}\frac{2-\sqrt{7-x}}{x-3}$

③ $\lim_{x\to 3^-}\frac{x-3}{|x-3|}$

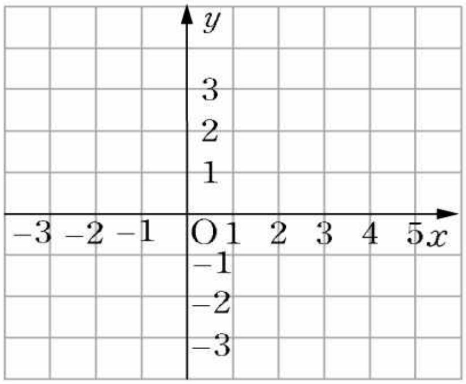
④ $\lim_{x\to\left(\frac{\pi}{2}\right)^+} e^{\tan x}$

⑤ $\lim_{x\to 0}\frac{\sin^3 2x}{x^3}$

⑥ $\lim_{x\to\infty}\frac{4x+1}{\sqrt{x^2-2x}}$

2. Sketch a graph of function $f(x)$ that satisfies all of the following conditions.

- $f(-1)=f(2)=-2$
- $\lim_{x\to-\infty} f(x)=0$
- $\lim_{x\to\infty} f(x)=1$
- $\lim_{x\to-1^+} f(x)=\infty$
- $\lim_{x\to-1^-} f(x)=\infty$
- $\lim_{x\to 2^-} f(x)=1,\quad \lim_{x\to 2^+} f(x)=-2$



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2. Integral Techniques

★ Integration by Parts

$$\int u v' dx = uv - \int u' v\, dx$$

◆ **Order of u and v'**

Logarithmic − **I**nverse Trig − **P**olynomial − **E**xponential − **T**rigonometric : **LIPET**

$$u \longleftrightarrow v'$$

[Proof] $\frac{d}{dx}(u\cdot v) = u'\cdot v + u\cdot v'$ (Product Rule)

Integrate both sides with respect to x:

$$\int \frac{d}{dx}(u\cdot v)\,dx = \int (u'\cdot v)\,dx + \int (u\cdot v')dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\therefore \underline{\int uv'\,dx = uv - \int u'v\,dx}$$

(Example 1) Evaluate the following indefinite integrals.

① $\int x e^{2x}\,dx$

$u =$ **Polynomial**, $v' =$ **Exponential**

$$\boxed{\begin{array}{ll} u = x, & v' = e^{2x} \\ u' = 1, & v = \frac{1}{2}e^{2x} \end{array}}$$

$$= \underset{\nwarrow\; uv}{x\cdot\frac{1}{2}e^{2x}} - \int \underset{\nwarrow\; u'v}{\frac{1}{2}e^{2x}}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

② $\int x\cos 3x\,dx$

$u =$ **Polynomial**, $v' =$ **Trigonometric**

$$\boxed{\begin{array}{ll} u = x, & v' = \cos 3x \\ u' = 1, & v = \frac{1}{3}\sin 3x \end{array}}$$

$$= \underset{\nwarrow\; uv}{x\cdot\frac{1}{3}\sin 3x} - \int \underset{\nwarrow\; u'v}{\frac{1}{3}\sin 3x}\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

③ $\int x^3 \ln x\,dx$

$u =$ **Logarithmic**, $v' =$ **Polynomial**

$$\boxed{\begin{array}{ll} u = \ln x, & v' = x^3 \\ u' = \frac{1}{x}, & v = \frac{1}{4}x^4 \end{array}}$$

$$= \underset{\nwarrow\; uv}{\frac{1}{4}x^4\cdot\ln x} - \int \underset{\nwarrow\; u'v}{\frac{1}{4}x^3}\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

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⑤ $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}}$

(Example 2) Evaluate $\displaystyle\int \sqrt{4-x^2}\,dx$.

Let $x = 2\sin\theta$.

$\dfrac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$\boxed{\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta}$

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$$= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int \sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta$$

$$= \int \sqrt{4\cos^2\theta}\cdot 2\cos\theta\,d\theta$$

$$= \int 4\cos^2\theta\,d\theta$$

$$= \int 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

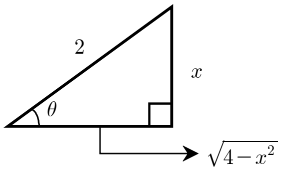
$$= \int (2 + 2\cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C$$

The answer should be expressed in terms of x.

$$x = 2\sin\theta \quad\Rightarrow\quad \sin\theta = \frac{x}{2} = \frac{opp}{hyp}$$



$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

$$2\sin\theta\cos\theta = 2\cdot\frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2} = \frac{x\sqrt{4-x^2}}{2}$$

$$\therefore \int \sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

[EX2] Evaluate the following integrals.

① $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

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★ Integration by Partial Fractions

◆ $\int \frac{N(x)}{D(x)}\,dx$ where $N(x)$ and $D(x)$ are polynomials.

If the degree of the numerator is **greater than or equal to** the degree of the denominator, use **long division** and write a **division statement**. Then integrate.

(Example 1) Evaluate the following integrals.

◆ Division Statement

$$\frac{N}{D} = Q + \frac{R}{D}$$

① $\int \frac{7x+4}{x+1}\,dx$

$$\begin{array}{rl} & \phantom{7x+{}}7 \\ x+1 & \overline{\big)\,7x+4} \\ - & \underline{\big|\,7x+7} \\ & \phantom{7x+{}}-3 \end{array}$$

$$\therefore\ \frac{7x+4}{x+1} = 7 - \frac{3}{x+1}$$

$$= \int \left(7 - \frac{3}{x+1}\right)dx$$

$$= 7x - 3\ln|x+1| + C$$

② $\int \frac{x^2+x+5}{x^2+4}\,dx$

$$\begin{array}{rl} & \phantom{x^2+x+{}}1 \\ x^2+4 & \overline{\big)\,x^2+x+5} \\ - & \underline{\big|\,x^2+4} \\ & \phantom{x^2+{}}x+1 \end{array}$$

$$\therefore\ \frac{x^2+x+5}{x^2+4} = 1 + \frac{x+1}{x^2+4}$$

$$= \int \left(1 + \frac{x+1}{x^2+4}\right)dx$$

$$= \int \left(1 + \frac{x}{x^2+4} + \frac{1}{x^2+4}\right)dx$$

$$= x + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

③ $\int \frac{x^5+2x^3+2}{x^2+1}\,dx$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\begin{array}{rl} & x^3+x \\ x^2+1 & \overline{\big)\,x^5+2x^3+0+0+2} \\ - & \underline{\big|\,x^5+x^3} \\ & x^3+2 \\ - & \underline{\big|\,x^3+x} \\ & -x+2 \end{array}$$

$$\therefore\ \frac{x^5+2x^3+2}{x^2+1} = x^3+x+\frac{-x+2}{x^2+1}$$

$$= \int \left(x^3+x+\frac{-x+2}{x^2+1}\right)dx$$

$$= \int \left(x^3+x-\frac{x}{x^2+1}+\frac{2}{x^2+1}\right)dx$$

$$= \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x + C$$

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[EX1] Evaluate the following integrals.

① $\int \frac{x^3+5x-4}{x-1}\,dx$

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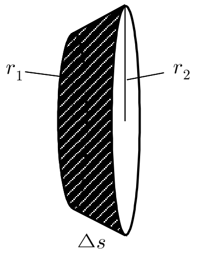
★ Surface Area of Solids of Revolution

The **area of the surface** generated by revolving a curve from $(a,\ c)$ to $(b,\ d)$ about the x**-axis** is defined by $S = \displaystyle\int 2\pi y\, ds$.

(1) $y = f(x),\quad a \le x \le b:\qquad S = \displaystyle\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

(2) $x = g(y),\quad c \le y \le d:\qquad S = \displaystyle\int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

[Proof]



Lateral Surface Area of a Frustum

$= \pi\,($*smaller radius* $+$ *larger radius*$)\,($*slant length*$)$

$= \pi(r_1 + r_2)\triangle s$

If $\triangle s \to 0,\ r_1 \approx r_2$.

Let $r_1 = r_2 = r$.

Lateral surface area of a frustum $= \pi(r + r)ds\ = 2\pi r\, ds$

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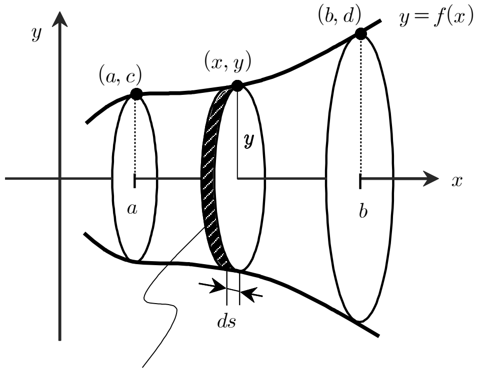
The curve $y = f(x)$ from $(a,\ c)$ to $(b,\ d)$ is revolved about the x**-axis**.



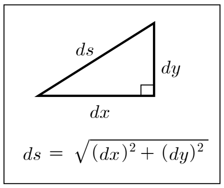
Lateral Surface Area$= 2\pi y\ ds$ (radius $= y$)

$$dS = 2\pi y\, ds$$

$$S = \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(or \quad \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \right)$$



(Example 1) Find the area of the surface generated by revolving the arc of $y = 2\sqrt{x}$ $(1 \le x \le 4)$ about the x-axis.

(Method 1) $y = 2\sqrt{x}$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^4 2\pi \cdot 2\sqrt{x} \cdot \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx$$

$$= \int_1^4 4\pi\sqrt{x}\sqrt{1 + \frac{1}{x}}\, dx = \int_1^4 4\pi\sqrt{x+1}\, dx$$

$$= \left[4\pi \cdot \frac{2}{3} \cdot (x+1)^{3/2}\right]_1^4$$

$$= \frac{8\pi}{3}\left[\ (x+1)^{3/2}\ \right]_1^4 = \frac{8\pi}{3}(5^{3/2} - 2^{3/2})$$

(Method 2) $y = 2\sqrt{x} \ \Rightarrow\ \dfrac{y}{2} = \sqrt{x} \ \Rightarrow\ x = \dfrac{y^2}{4} \ \Rightarrow\ \dfrac{dx}{dy} = \dfrac{y}{2}$

If $x = 1,\ y = 2$. **If** $x = 4,\ y = 4$.

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_2^4 2\pi y \sqrt{1 + \left(\frac{y}{2}\right)^2}\, dy$$

$$= \int_2^4 2\pi y \sqrt{1 + \frac{y^2}{4}}\, dy = \int_2^4 \pi y \sqrt{4 + y^2}\, dy$$

Let $u = 4 + y^2$.

$\dfrac{du}{dy} = 2y$

$\dfrac{1}{2}du = y\,dy$

If $y = 2,\ u = 8$.

If $y = 4,\ u = 20$.

$$= \int_8^{20} \frac{\pi}{2}\sqrt{u}\, du$$

$$= \left[\frac{\pi}{2} \cdot \frac{2}{3} \cdot u^{3/2}\right]_8^{20} = \frac{\pi}{3}\left[\ u^{3/2}\ \right]_8^{20}$$

$$= \frac{\pi}{3}\left[\ 20^{3/2} - 8^{3/2}\right]$$

$$\left(or \ = \frac{8\pi}{3}\left(5^{\ 3/2} - 2^{3/2}\right) \right)$$

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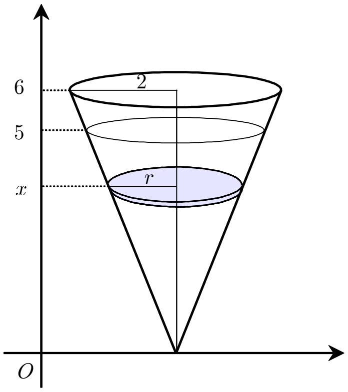
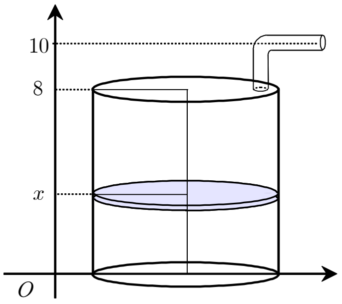
[EX3] An inverted conical tank with a height of $6\,m$ and a diameter of $4\,m$ is filled with water to a height of $5\,m$. Find the work required, in J, to empty the tank by pumping all the water to the top rim of the tank. The density of water is $1000\,kg/m^3$ and the acceleration of gravity is $g=9.8\,m/s^2$.



[EX4] A cylindrical tank with a height of $8\,ft$ and a diameter of $8\,ft$ is half-full of oil. Find the work required, in ft-lb, to empty the tank by pumping all the oil to an outlet pipe $2\,ft$ above the top of the tank. The density of oil is $40\,lb/ft^3$.

Foot-pound force (*ft-lb* or *ft-lb$_f$*) is a unit of work or energy in the imperial system of units.



4. Parametric Equations and Polar Coordinates

★ Parametric Equations

The x and y coordinates of a point on a curve can be expressed as functions $x=f(t)$, $y=g(t)$ of a third variable t, called a **parameter**. Such equations are called **parametric equations**.

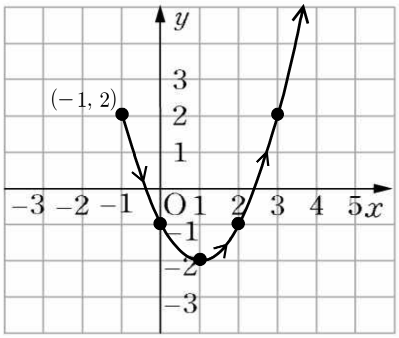
$$x=f(t),\quad y=g(t)$$

(Example 1) (1) Sketch the curve by plotting points. Indicate the direction in which the curve is traced.
(2) Eliminate the parameter to find the Cartesian equation of the curve.
(3) Check your sketch with a graphing calculator.

① $x=t+1,\quad y=t^2-2\quad (t\geq -2)$

(1)

t	x	y
-2	-1	2
-1	0	-1
0	1	-2
1	2	-1
2	3	2
⋮	⋮	⋮

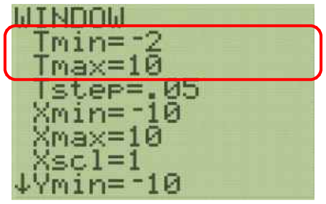


(2) $x=t+1 \quad\Rightarrow\quad t=x-1$

Since $t\geq -2$, $x\geq -1$.

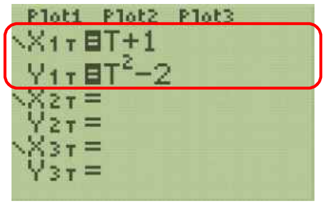
Substitute $t=x-1$ into $y=t^2-2$. $\qquad\therefore\ y=(x-1)^2-2\quad (x\geq -1)$

(3) Set the **MODE** to **PAR**. Then set the **WINDOW** to Tmin$=-2$, Tmax$=10$.

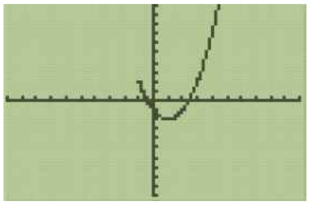


WINDOW
Tmin=-2
Tmax=10
Tstep=.05
Xmin=-10
Xmax=10
Xscl=1
↓Ymin=-10

Press **Y=**. Enter $t+1$ into X_{1T}, and t^2-2 into Y_{1T}. Then press **GRAPH**.



Plot1 Plot2 Plot3
\X1T=T+1
Y1T=T²-2
\X2T=
Y2T=
\X3T=
Y3T=



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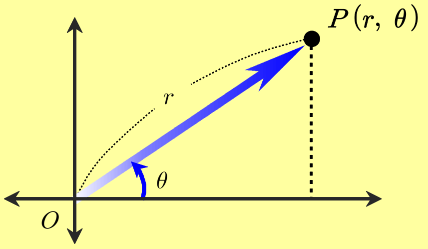
★ Polar Coordinates

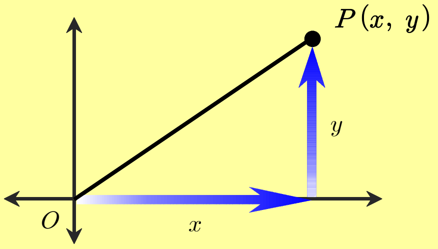
The polar coordinate system locates a point (P) in a plane by its **distance** (r) from the origin and by the **angle** (θ) between the positive x-axis and the line segment connecting the origin to the point P.

- **Polar coordinate system**



(e.g.) $P\left(4,\ \frac{\pi}{6}\right) \ \Rightarrow\ r=4$ and $\theta=\frac{\pi}{6}$

- **Cartesian coordinate system**



(e.g.) $P(2\sqrt{3},\ 2) \ \Rightarrow\ x=2\sqrt{3}$ and $y=2$

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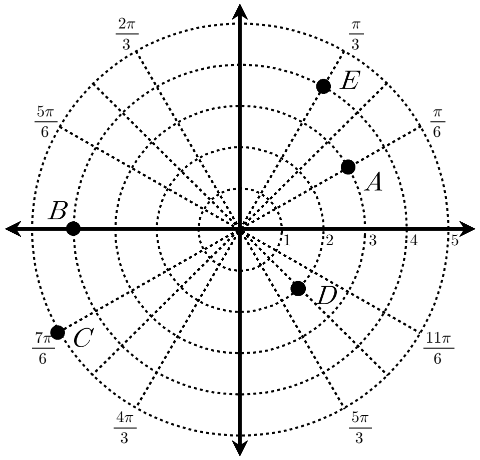
(Example 1) Plot the points in the polar coordinate system.

① $A(3,\ 30°)$ ② $B(4,\ \pi)$ ③ $C(-5,\ 30°)$ ④ $D\left(2,\ -\frac{\pi}{4}\right)$ ⑤ $E\left(-4,\ -\frac{2\pi}{3}\right)$



Note: If $r<0$, then reflect the point about the origin. $(-k,\ \theta) = (k,\ \theta+\pi)$

① $A(3,\ 30°)$: $r=3$ and $\theta=\frac{\pi}{6}$

② $B(4,\ \pi)$: $r=4$ and $\theta=\pi$

③ $C(-5,\ 30°) = C(5,\ 210°)$: $r=5$ and $\theta=\frac{7\pi}{6}$

④ $D\left(2,\ -\frac{\pi}{4}\right)$: $r=2$ and $\theta=-\frac{\pi}{4}$

⑤ $E\left(-4,\ -\frac{2\pi}{3}\right) = E\left(4,\ \frac{\pi}{3}\right)$: $r=4$ and $\theta=\frac{\pi}{3}$

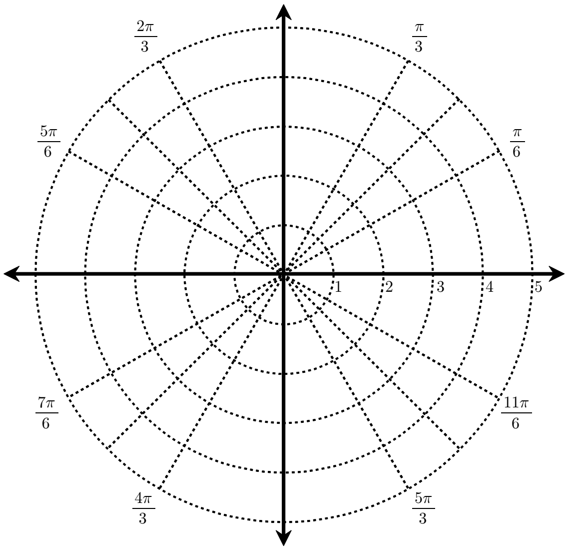
[EX1] Plot the following points in the polar coordinate system.

① $A(2,\ 0)$ ② $B(4,\ 150°)$

③ $C\left(5,\ \frac{5\pi}{4}\right)$ ④ $D\left(3,\ \frac{3\pi}{2}\right)$

⑤ $E(-5,\ 180°)$ ⑥ $F\left(1,\ -\frac{3\pi}{2}\right)$

⑦ $G\left(-3,\ \frac{2\pi}{3}\right)$ ⑧ $H\left(-4,\ -\frac{3\pi}{4}\right)$



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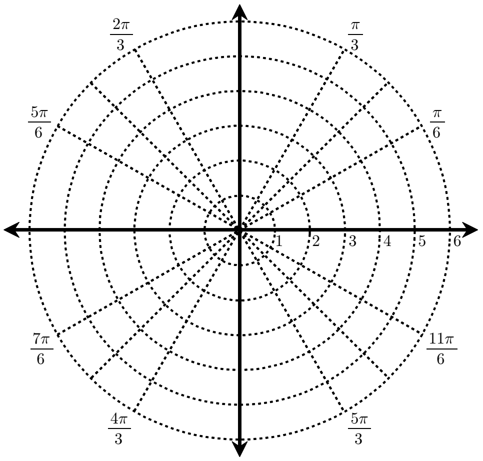
④ $r = 2 - 4\sin\theta$

θ	r



◆ Rose Curves

$r = a\sin n\theta$, $r = a\cos n\theta$

(Example 3) Consider the polar curve $r = 4\cos 3\theta$.

(1) Sketch the curve by plotting points. Indicate the direction in which the curve is traced.

(2) Check your sketch with a graphing calculator.

(1)

θ	r
0	4
$\frac{\pi}{6}$	0
$\frac{\pi}{3}$	-4
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	4
$\frac{5\pi}{6}$	0
π	-4
$\frac{7\pi}{6}$	0
$\vdots$	$\vdots$

$r = 0$ when $\cos 3\theta = 0$.

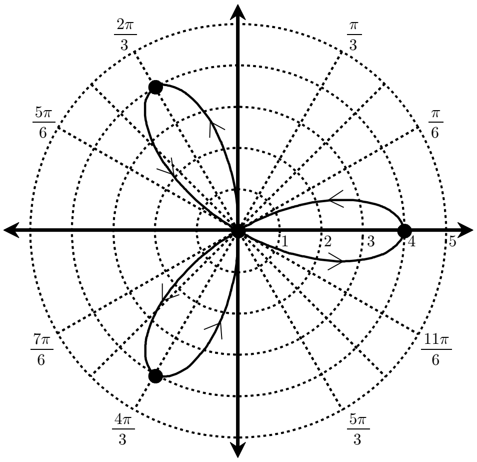
$$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \cdots$$

$$\therefore\ \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \cdots$$

r has a maximum/minimum when $\cos 3\theta = \pm 1$.

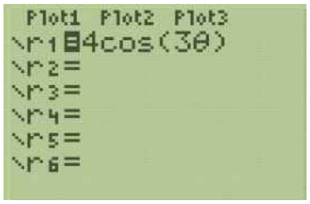
$$3\theta = 0, \pi, 2\pi, 3\pi, \cdots$$

$$\therefore\ \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \cdots$$

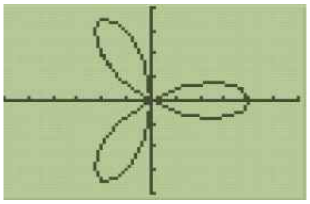


(2)

```
NORMAL  SCI  ENG
FLOAT  0123456789
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
↓NEXT↓
```



```
Plot1  Plot2  Plot3
\r1 ■4cos(3θ)
\r2=
\r3=
\r4=
\r5=
\r6=
```



[-6, 6] × [-4, 4]

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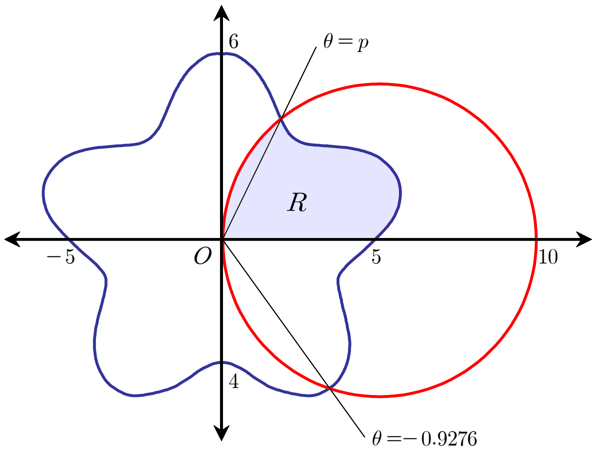
② common to the Limaçon $r=5+4\sin\theta$ and the curve $r=4-\sin(4\theta)$.

[EX8] The graphs of the polar curves $r=f(\theta)=5+\sin(5\theta)$ and $r=g(\theta)=10\cos(\theta)$ are shown below. The curves intersect when $\theta=-0.9276$ and $\theta=p$. Use a graphing calculator to solve the following questions.

① Find the value of p.

② Let R be the region enclosed by the curves $r=f(\theta)$, $r=g(\theta)$, and the x-axis. Find the area of R.



③ The ray $\theta=k\left(0<k<\dfrac{\pi}{2}\right)$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value k.

④ Let $w(\theta)$ be the distance between the curves $r=f(\theta)$ and $r=g(\theta)$ at angle θ, $-0.9276\le\theta\le p$. Find the average distance of $w(\theta)$ over the interval $-0.9276\le\theta\le p$.

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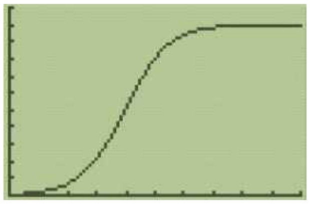
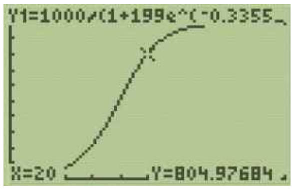
③ Using a graphing calculator, sketch the solution curve.



[0, 40] × [0, 1100]

```
Plot1  Plot2  Plot3
\Y1 ■        1000
       1+199e^-0.33555X
\Y2=
\Y3=
\Y4=
\Y5=
```

④ Find the number of people who have heard the rumor after 20 days.



```
Y1=1000/(1+199e^(-0.3355…
X=20          Y=804.97684
```

$\therefore\ 805$ people

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⑤ Show that the rumor spreads the fastest when it reaches half the carrying capacity.

The rumor spreads the fastest when $\frac{dy}{dt}$ reaches its maximum.

$$\frac{dy}{dt}=ky\left(1-\frac{y}{1000}\right) \Rightarrow \frac{dy}{dt}=ky-\frac{ky^2}{1000}$$

Find the second derivative.

$$\frac{d^2y}{dt^2}=k\frac{dy}{dt}-\frac{2ky}{1000}\frac{dy}{dt}$$

$$=k\frac{dy}{dt}\left(1-\frac{y}{500}\right)$$

Let $\frac{d^2y}{dt^2}=0.$ ⟶ Since $k>0$ and $\frac{dy}{dt}>0,\ 1-\frac{y}{500}=0.$

$\therefore\ y=500$

If $y<500,\ \frac{d^2y}{dt^2}>0.$ (The graph of y is concave up)

If $y>500,\ \frac{d^2y}{dt^2}<0.$ (The graph of y is concave down)

$\therefore\ \frac{dy}{dt}$ is the greatest when $y=500.$

[EX1] Solve the following differential equations.

$$\frac{dy}{dt}=ky\left(1-\frac{y}{L}\right) \Rightarrow y=\frac{L}{1+Ae^{-kt}}$$

① $\frac{dy}{dt}=2y\left(1-\frac{y}{500}\right),\quad y(0)=10$

② $\frac{dy}{dt}=\frac{3y}{10}-\frac{y^2}{1000},\quad y(0)=5$

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Chapter Review Exercises

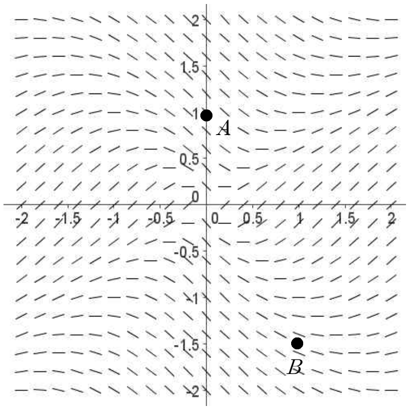
The use of a calculator is permitted.

1. The slope field for the differential equation $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$ is shown below.

① Sketch the solution curve that passes through the indicated point.

(1) $A\,(0,\,1\,)$

(2) $B\,(1\,,-1.5\,)$

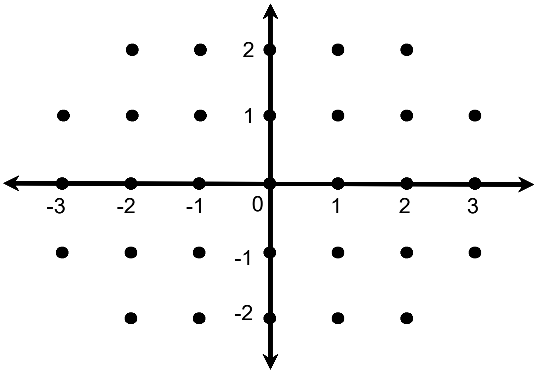


② Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 2$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(0.5)$.

2. Consider the differential equation $\frac{dy}{dx} = -\frac{xy}{2}$ with the initial condition $f(1) = 1$.

① Sketch a slope field for the differential equation at the indicated points.

② Use the slope field to sketch the solution curve that passes through the point $(1,\,1)$.



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★ Convergence Tests

[1] Divergence Test (*n*th Term Test)

If $\lim\limits_{n\to\infty} a_n \neq 0$, **then** $\sum\limits_{n=1}^{\infty} a_n$ **diverges.**

(Contrapositive statement: If $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.)

Note: If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ may either converge or diverge.

(Example 1) Show that the series $\sum\limits_{n=1}^{\infty} \frac{n}{n+2} = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + ...$ diverges.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{n}{n+2} = \lim\limits_{n\to+\infty} \frac{1}{1+\frac{2}{n}} = 1 \neq 0$ ∴ The series diverges by the **Divergence Test**.

(Example 2) Disprove the following statement: If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Consider the **harmonic series**, $\sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{1}{n} = 0$, but the series diverges.

(Example 3) Disprove the following statement: If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Consider the **geometric series** with $r = \frac{1}{2}$, $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.

$\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} \frac{1}{2^{n-1}} = 0$, but the series converges.

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[EX1] Determine whether the series converges or diverges.

① $\frac{1}{3} + \frac{3}{7} + \frac{5}{11} + \frac{7}{15} + \cdots$

② $\frac{1}{3} + \frac{7}{9} + \frac{25}{27} + \frac{79}{81} + \cdots$

③ $\sum\limits_{n=2}^{\infty} \frac{n}{\ln n}$

④ $\sum\limits_{n=1}^{\infty} \frac{n^2+1}{2n^2+n+5}$

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[8] Alternating Series Test

A series whose terms are alternately positive and negative is called an **alternating series**.

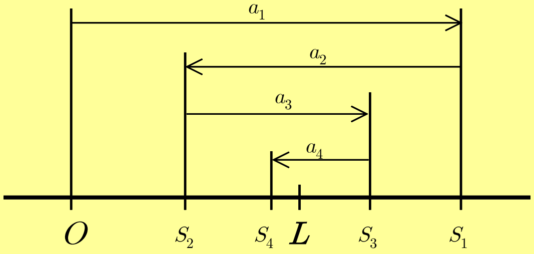
$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$$

If **1.** $\lim_{n\to\infty} a_n = 0$ and,

2. $a_{n+1} < a_n$ for all n,

then **the alternating series converges**.



$$\boxed{\lim_{n\to\infty} S_n = L}$$

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(Example 11) Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+1}$ converges or diverges.

It is an alternating series with $a_n = \frac{1}{\sqrt{n}+1}$.

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = 0$

2. $\frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1}$

$\therefore$ The given series converges by the **Alternating Series Test**.

[EX10] Determine whether the alternating series converges or diverges.

① $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

② $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

③ $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n+2}$

④ $\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{\ln n}$

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② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$, $error < 0.001$

③ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$, $error < 0.005$

④ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$, $error < 0.0001$

(Example 2) Approximate the sum of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^n}$ to three decimal places.

Step 1. It is an alternating series with $a_n = \frac{1}{(n+1)^n}$.

$$a_{n+1} = \frac{1}{(n+2)^{n+1}} < 0.0005$$

$n=2:$ $a_3 = \frac{1}{4^3} = 0.0156 < 0.0005$ ✗

$n=3:$ $a_4 = \frac{1}{5^4} = 0.0016 < 0.0005$ ✗

$n=4:$ $a_5 = \frac{1}{6^5} = 0.0001286 < 0.0005$ ✓

∴ At least 4 terms are needed.

Note:
To be accurate to three decimal places, the error should be less than 0.0005.

Step 2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^n} \approx S_4 = \frac{1}{2^1} - \frac{1}{3^2} + \frac{1}{4^3} - \frac{1}{5^4} \approx 0.403$

$\therefore\ 0.403$

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7. Infinite Sequences and Series, Part Ⅱ

★ Power Series

◆ **Power Series about $x=0$ (or centered at $x=0$)**

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \qquad (a_n\text{: coefficients})$$

◆ **Power Series about $x=c$ (or centered at $x=c$)**

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

◆ **Radius of Convergence**

There is a number R such that a power series **converges** if $|x-c|<R$ and diverges if $|x-c|>R$. The number R is called the radius of convergence of the power series.

◆ **Interval of Convergence**

The interval of convergence is the set of values of x for which a power series converges.

$$|x-c|<R \quad \Rightarrow \quad -R<x-c<R \quad \Rightarrow \quad c-R<x<c+R$$

Note that the power series may converge or diverge at the endpoints of the interval. Thus **each endpoint must be tested separately** for convergence or divergence.

(Example 1) Find the radius and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+2)^n}{\ln n}$.

Use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(2x+2)^{n+1}}}{\ln(n+1)}\cdot\frac{\ln n}{\cancel{(2x+2)^n}}\right|$$

$$= \lim_{n\to\infty}\left|(2x+2)\cdot\frac{\ln n}{\ln(n+1)}\right|$$

$$= \left|(2x+2)\cdot 1\right| = |2x+2| = 2|x+1|$$

The series converges if $2|x+1| < 1$.

$$|x+1| < \frac{1}{2} \qquad \therefore \text{ The } \textbf{radius of convergence} \text{ is } \frac{1}{2}.$$

$$|x+1| < \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2} < x+1 < \frac{1}{2} \quad \Rightarrow \quad -\frac{3}{2} < x < -\frac{1}{2}$$

Endpoints:

If $x=-\dfrac{3}{2}$, $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+2)^n}{\ln n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\ln n}$

1. $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\ln n} = 0$

2. $\dfrac{1}{\ln(n+1)} < \dfrac{1}{\ln n}$

∴ The series converges by the Alternating Series Test.

If $x=-\dfrac{1}{2}$, $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+2)^n}{\ln n} = \sum_{n=1}^{\infty}\frac{1}{\ln n}$

$\ln n < n$

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{\ln n} > \sum_{n=1}^{\infty}\frac{1}{n}$ which is the divergent harmonic series.

∴ The series diverges by the Direct Comparison Test.

∴ The **interval of convergence** is $-\dfrac{3}{2} \le x < -\dfrac{1}{2}$.

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(Example 4) Find the <u>first three non-zero terms</u> of the Maclaurin series for $f(x)=e^x\sin x$.

(Method 1)

$$f(x)=e^x\sin x \qquad \Rightarrow\ f(0)=0$$

$$f'(x)=e^x\sin x+e^x\cos x=e^x(\sin x+\cos x) \qquad \Rightarrow\ f'(0)=1$$

$$f''(x)=e^x(\sin x+\cos x)+e^x(\cos x-\sin x)=2e^x\cos x \qquad \Rightarrow\ f''(0)=2$$

$$f'''(x)=2e^x\cos x+2e^x(-\sin x) \qquad \Rightarrow\ f'''(0)=2$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^2+\cdots$$

By substitution, $\underline{e^x\sin x=x+x^2+\frac{1}{3}x^3+\cdots}$

(Method 2) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ and $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

Multiply the two Maclaurin Series.

$$e^x\sin x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

$$=x+x^2-\frac{x^3}{3!}+\frac{x^3}{2!}-\frac{x^4}{3!}+\frac{x^4}{3!}+\cdots$$

$$\therefore\ \underline{e^x\sin x=x+x^2+\frac{x^3}{3}+\cdots}$$

[EX5] Find the <u>first three non-zero terms</u> of the Maclaurin series for the function f.

① $f(x)=\sqrt{x+1}$

② $f(x)=e^x\cos x$

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★ Taylor Series

◆ Taylor Series

Let f be a function that is infinitely differentiable at $x = a$.

Then the Taylor series for f about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

[Proof] Let $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n + \cdots$ where $|x-a| < R$.

$f(a) = c_0$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1} + \cdots \quad \Rightarrow \quad f'(a) = c_1 \quad \therefore\ c_1 = f'(a)$

$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots + n(n-1)c_n(x-a)^{n-2} + \cdots \quad \Rightarrow \quad f''(a) = 2c_2 = 2!c_2 \quad \therefore\ c_2 = \frac{f''(a)}{2!}$

$f'''(x) = 6c_3 + 24c_4(x-a) + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3} + \cdots \quad \Rightarrow \quad f'''(a) = 6c_3 = 3!c_3 \quad \therefore\ c_3 = \frac{f'''(a)}{3!}$

$f^{(4)}(x) = 24c_4 + 120c_5(x-a) + \cdots + n(n-1)(n-2)(n-3)c_n(x-a)^{n-4} + \cdots \quad \Rightarrow \quad f^{(4)}(a) = 24c_4 = 4!c_4 \quad \therefore\ c_4 = \frac{f^{(4)}(a)}{4!}$

$\vdots \qquad \vdots \qquad \vdots$

$f^{(n)}(x) = n!c_n + (n+1)!c_{n+1}(x-a) + \cdots \quad \Rightarrow \quad f^{(n)}(a) = n!c_n \quad \therefore\ c_n = \frac{f^{(n)}(a)}{n!}$

(Example 1) Find the Taylor series and the interval of convergence for $f(x) = \sin x$ about $x = \pi$.

$f(x) = \sin x \quad \Rightarrow \quad f(\pi) = 0$

$f'(x) = \cos x \quad \Rightarrow \quad f'(\pi) = -1$

$f''(x) = -\sin x \quad \Rightarrow \quad f''(\pi) = 0$

$f'''(x) = -\cos x \quad \Rightarrow \quad f'''(\pi) = 1$

$f^{(4)}(x) = \sin x \quad \Rightarrow \quad f^{(4)}(\pi) = 0$

$f^{(5)}(x) = \cos x \quad \Rightarrow \quad f^{(5)}(\pi) = -1$

$\vdots \qquad \vdots$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

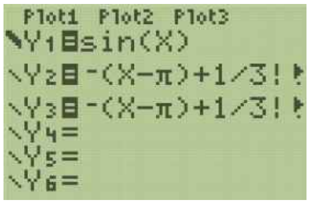
By substitution,

$$\sin x = 0 - (x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 + 0 - \frac{1}{5!}(x-\pi)^5 + \cdots$$

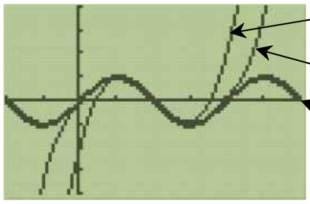
$\therefore$ The Taylor series for $\sin x$ about $x = \pi$ is

$$\sin x = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots + \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$



```
Plot1 Plot2 Plot3
\Y1■sin(X)
\Y2■-(X-π)+1/3!▸
\Y3■-(X-π)+1/3!▸
\Y4=
\Y5=
\Y6=
```



$y_2 = -(x-\pi) + \frac{1}{3!}(x-\pi)^3$

$y_3 = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7$

$y_1 = \sin x$

$[-\pi,\ 3\pi] \times [-4,\ 4]$

To find the interval of convergence, use the **ratio test**.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-\pi)^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(x-\pi)^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-\pi)^2}{(2n+3)(2n+2)}\right| = 0 < 1$$

Note $|(-1)^N| = 1$

The series converges for $x \in R$.

$\therefore$ The **interval of convergence** is **all real numbers**.

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③ Approximate $g(0.3)$ such that this approximation differs from the actual value of $g(0.3)$ by less than 0.002.

④ Find the interval of convergence of the series for the function $g(x)$.

15. Let f be the function defined by $f(x) = \cos(2x)$.

① Use the second-degree Taylor polynomial, $P_2(x)$, for f about $x = \frac{\pi}{8}$ to approximate $f\left(\frac{\pi}{6}\right)$.

② Use the Lagrange error bound to show that $\left|f\left(\frac{\pi}{6}\right) - P_2\left(\frac{\pi}{6}\right)\right| < 0.005$.

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SOLUTIONS

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[EX2]

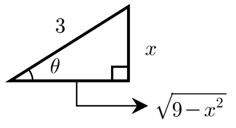
① $Let\ x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}}$$
$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9(1-\sin^2\theta)}}$$
$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{3\cos\theta}$$
$$= \int 9\sin^2\theta\, d\theta$$
$$= \int 9\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 9\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) + C$$
$$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C$$
$$= \frac{9}{2}\theta - \frac{9}{4}\cdot 2\sin\theta\cos\theta + C$$
$$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$
$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$

$x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$



$\therefore\ \cos\theta = \frac{\sqrt{9-x^2}}{3}$

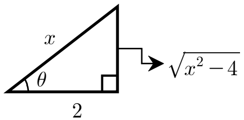
② $Let\ x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\, d\theta$

$$= \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^3\theta\cdot\sqrt{4\sec^2\theta - 4}}$$
$$= \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^3\theta\cdot\sqrt{4(\sec^2\theta - 1)}}$$
$$= \int \frac{\cancel{2}\sec\theta\,\cancel{\tan\theta\, d\theta}}{8\sec^3\theta\cdot\cancel{2}\cancel{\tan\theta}}$$
$$= \int \frac{d\theta}{8\sec^2\theta}$$
$$= \int \frac{1}{8}\cos^2\theta\, d\theta$$
$$= \int \frac{1}{8}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{8}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) + C$$
$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C$$
$$= \frac{1}{16}\theta + \frac{1}{32}\cdot 2\sin\theta\cos\theta + C$$
$$= \frac{1}{16}\sec^{-1}\left(\frac{x}{2}\right) + \frac{1}{16}\cdot\frac{\sqrt{x^2-4}}{x}\cdot\frac{2}{x} + C$$
$$= \frac{1}{16}\sec^{-1}\left(\frac{x}{2}\right) + \frac{\sqrt{x^2-4}}{8x^2} + C$$

$x = 2\sec\theta \Rightarrow \sec\theta = \frac{x}{2}$



$\therefore\ \sin\theta = \frac{\sqrt{x^2-4}}{x}$

$\cos\theta = \frac{2}{x}$

③ $Let\ x = 2\tan\theta$

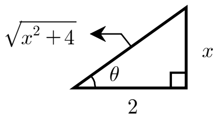
$dx = 2\sec^2\theta\, d\theta$

$$= \int \sqrt{4\tan^2\theta + 4}\cdot 2\sec^2\theta\, d\theta = \int \sqrt{4(\tan^2\theta + 1)}\cdot 2\sec^2\theta\, d\theta = \int 2\sec\theta\cdot 2\sec^2\theta\, d\theta = \int 4\sec^3\theta\, d\theta$$
$$= 4\int \sec^3\theta\, d\theta$$

Refer to p63 [EX6]

$$= 4\cdot\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C$$
$$= 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| + C$$
$$= 2\cdot\frac{\sqrt{x^2+4}}{2}\cdot\frac{x}{2} + 2\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$
$$= \frac{x\sqrt{x^2+4}}{2} + 2\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$
$$\left(or \quad = \frac{x\sqrt{x^2+4}}{2} + 2\ln\left|\sqrt{x^2+4} + x\right| + C \right)$$

$x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$



$\therefore\ \sec\theta = \frac{\sqrt{x^2+4}}{2}$

[EX3]

① $Let\ x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$if\ x = 0,\ \ 0 = 4\tan\theta \ \ \therefore \theta = 0$

$if\ x = 4\sqrt{3},\ \ 4\sqrt{3} = 4\tan\theta \ \ \therefore \theta = \frac{\pi}{3}$

$$= \int_0^{\frac{\pi}{3}} \frac{4\sec^2\theta\cdot d\theta}{\sqrt{16 + 16\tan^2\theta}} = \int_0^{\frac{\pi}{3}} \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int_0^{\frac{\pi}{3}} \sec\theta\, d\theta$$
$$= \Big[\ \ln|\sec\theta + \tan\theta|\ \Big]_0^{\frac{\pi}{3}} = \ln(2 + \sqrt{3})$$

② $Let\ x = \sin\theta$

$dx = \cos\theta\, d\theta$

$if\ x = 0,\ \ 0 = \sin\theta \ \ \therefore \theta = 0$

$if\ x = \frac{1}{2},\ \ \frac{1}{2} = \sin\theta \ \ \therefore \theta = \frac{\pi}{6}$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos\theta\, d\theta}{\left(\sqrt{1-\sin^2\theta}\right)^3} = \int_0^{\frac{\pi}{6}} \frac{\cos\theta\, d\theta}{\cos^3\theta} = \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2\theta}\, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sec^2\theta\, d\theta = \Big[\ \tan\theta\ \Big]_0^{\frac{\pi}{6}} = \frac{1}{\sqrt{3}}$$

③ $Let\ x = \sec\theta$ $\quad if\ x = \sqrt{2},\ \sqrt{2} = \sec\theta \ \ \therefore\ \theta = \frac{\pi}{4}$

$dx = \sec\theta\tan\theta\, d\theta$ $\quad if\ x = 2,\ \ 2 = \sec\theta \ \ \therefore\ \theta = \frac{\pi}{3}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta\, d\theta}{\sec^4\theta\cdot\sqrt{\sec^2\theta - 1}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta\, d\theta}{\sec^4\theta\cdot\tan\theta}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^3\theta\, d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\cdot\cos\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 - \sin^2\theta)\cdot\cos\theta\, d\theta$$

$Let\ \ u = \sin\theta$ $\quad if\ \theta = \frac{\pi}{4},\ \ u = \frac{1}{\sqrt{2}}$

$du = \cos\theta\, d\theta$ $\quad if\ \ \theta = \frac{\pi}{3},\ \ u = \frac{\sqrt{3}}{2}$

$$= \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} (1 - u^2)\, du = \left[u - \frac{1}{3}u^3\right]_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}}$$
$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{3}\cdot\frac{3\sqrt{3}}{8}\right) - \left(\frac{1}{\sqrt{2}} - \frac{1}{3\cdot 2\sqrt{2}}\right)$$
$$= \frac{3\sqrt{3}}{8} - \frac{5}{6\sqrt{2}} = \frac{3\sqrt{3}}{8} - \frac{5\sqrt{2}}{12}$$

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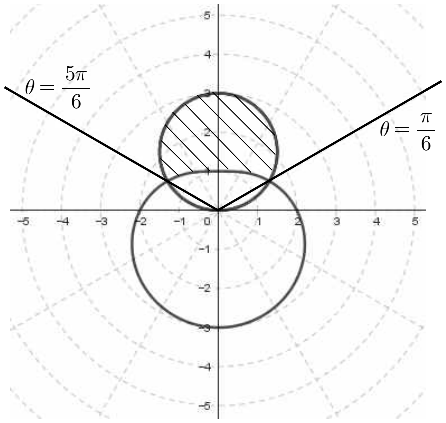
②



$r = 3\sin\theta$

θ	r
0	0
$\pi/6$	1.5
$\pi/2$	3
$5\pi/6$	1.5
π	0

$r = 2 - \sin\theta$

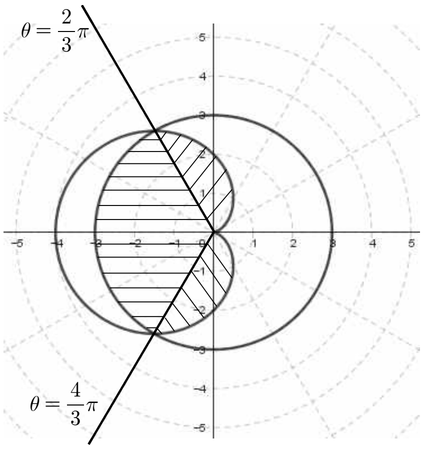
θ	r
0	2
$\pi/2$	1
π	2
$3\pi/2$	3

Intersection :

$$3\sin\theta = 2 - \sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned}
Area &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(R^2 - r^2)d\theta \\
&= 2\cdot\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(R^2 - r^2)d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[(3\sin\theta)^2 - (2 - \sin\theta)^2\right]d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[9\sin^2\theta - (4 - 4\sin\theta + \sin^2\theta)\right]d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[8\sin^2\theta - 4 + 4\sin\theta\right]d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) - 4 + 4\sin\theta\right]d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left[-4\cos 2\theta + 4\sin\theta\right]d\theta \\
&= \Big[-2\sin 2\theta - 4\cos\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= 0 - \left(-2\cdot\frac{\sqrt{3}}{2} - 4\cdot\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}
\end{aligned}$$

③



$r = 2 - 2\cos\theta$

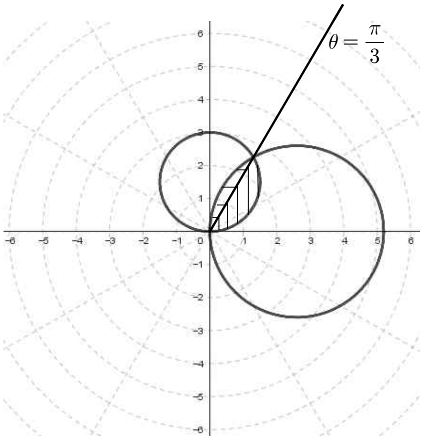
θ	r
0	0
$\pi/2$	2
π	4
$3\pi/2$	2
2π	0

Intersection :

$$2 - 2\cos\theta = 3$$
$$-2\cos\theta = 1$$
$$\cos\theta = -\frac{1}{2}$$
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\begin{aligned}
Area &= 2\cdot\frac{1}{2}\int_0^{\frac{2\pi}{3}}(2 - 2\cos\theta)^2 d\theta + \textit{Area of sector of the circle} \\
&= \int_0^{\frac{2\pi}{3}}(2 - 2\cos\theta)^2 d\theta + \frac{1}{2}(3)^2\cdot\frac{2\pi}{3} \\
&= \int_0^{\frac{2\pi}{3}}\left[4 - 8\cos\theta + 4\cos^2\theta\right]d\theta + 3\pi \\
&= \int_0^{\frac{2\pi}{3}}\left[4 - 8\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right]d\theta + 3\pi \\
&= \int_0^{\frac{2\pi}{3}}\left[6 - 8\cos\theta + 2\cos 2\theta\right]d\theta + 3\pi \\
&= \Big[6\theta - 8\sin\theta + \sin 2\theta\Big]_0^{\frac{2\pi}{3}} + 3\pi \\
&= \left(4\pi - 8\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - 0 + 3\pi = 7\pi - \frac{9\sqrt{3}}{2}
\end{aligned}$$

④



Intersection :

$$3\sin\theta = 3\sqrt{3}\cos\theta$$
$$\sin\theta = \sqrt{3}\cos\theta$$
$$\frac{\sin\theta}{\cos\theta} = \sqrt{3}$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \frac{\pi}{3}$$

$$\begin{aligned}
Area &= \int_0^{\frac{\pi}{3}}(3\sin\theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(3\sqrt{3}\cos\theta)^2 d\theta \\
&= \int_0^{\frac{\pi}{3}}9\sin^2\theta d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}27\cos^2\theta d\theta \\
&= \int_0^{\frac{\pi}{3}}9\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}27\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta \\
&= 9\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}} + 27\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= 9\left[\left(\frac{\pi}{6} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right)\right] + 27\left[\frac{\pi}{4} + 0 - \left(\frac{\pi}{6} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2}\right)\right] \\
&= \frac{3\pi}{2} - \frac{9\sqrt{3}}{8} + 27\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) \\
&= \frac{3\pi}{2} - \frac{9\sqrt{3}}{8} + \frac{9\pi}{4} - \frac{27\sqrt{3}}{8} = \frac{15\pi}{4} - \frac{9\sqrt{3}}{2}
\end{aligned}$$

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③ $\left| R_4(1) \right| = \left| g(1) - P_4(1) \right| < \left| \dfrac{(1)^6}{30} \right| = \dfrac{1}{30} < \dfrac{1}{20}$

④ $h(x) = (x-1) - \dfrac{(x-1)^2}{2!} + \dfrac{(x-1)^3}{3!} - \dfrac{(x-1)^4}{4!} + \cdots$
$= e^{x-1} - 1$

9.

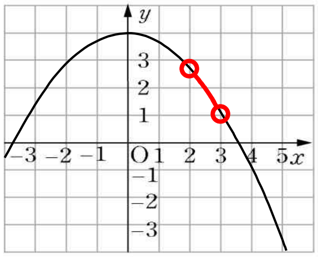
① $f(x) = (x-2) - \dfrac{(x-2)^2}{2\cdot 3^1} + \dfrac{(x-2)^3}{3\cdot 3^2} - \cdots + (-1)^n \dfrac{(x-2)^{n+1}}{(n+1)\cdot 3^n} + \cdots$

$f'(x) = 1 - \dfrac{(x-2)}{3} + \dfrac{(x-2)^2}{3^2} - \cdots + (-1)^n \dfrac{(x-2)^n}{3^n} + \cdots$

② f' *is an infinite geometric series with* $r = -\dfrac{x-2}{3}$.

$f'(x) = \dfrac{a}{1-r} = \dfrac{1}{1+\dfrac{x-2}{3}} = \dfrac{3}{3+(x-2)} = \dfrac{3}{x+1}$

10.



$f^{(6)}(x) < 3$ *for* $2 < x < 3$.

$\left| f(3) - P_5(3) \right| \le \left| \dfrac{f^{(6)}(c)}{6!}(3-2)^6 \right| < \dfrac{3}{720} = \dfrac{1}{240} < \dfrac{1}{200}$

11. $f^{(4)}(x) = e^x \quad \Rightarrow \quad e^x < e^{0.5}$ *for* $0 < x < 0.5$

$\left| R_3(0.5) \right| = \left| f(0.5) - P_3(0.5) \right| \le \left| \dfrac{f^{(4)}(c)}{4!}(0.5-0)^4 \right| < \left| \dfrac{e^{0.5}}{4!}(0.5)^4 \right| = 0.0043$

12. $\left| R_3(0.5) \right| = \left| f(0.5) - P_3(0.5) \right| < \left| \dfrac{(0.5)^5}{5!} \right| = 0.00026$

13. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$

$e^{-x^2} = 1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \cdots + (-1)^n \cdot \dfrac{x^{2n}}{n!} + \cdots$

$$\int_0^{0.5} e^{-x^2} dx = \int_0^{0.5} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right] dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)\cdot n!} + \cdots \right]_0^{0.5}$$

$$= (0.5) - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5\cdot 2!} - \frac{(0.5)^7}{7\cdot 3!} + \cdots$$

The series is a convergent alternating series.

To be accurate to three decimal places, the error should be less than 0.0005.

$|R_n| < |a_{n+1}| < 0.0005 \quad \Rightarrow \quad \dfrac{(0.5)^5}{5\cdot 2!} = 0.003$ ✗ $\qquad \dfrac{(0.5)^7}{7\cdot 3!} = 0.00018$ ✓

$\therefore \displaystyle\int_0^{0.5} e^{-x^2} dx \approx (0.5) - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5\cdot 2!} = 0.461$

14.

① $f(x) = 1 - 2x + \dfrac{\frac{3!}{\ln 2}}{2!}x^2 - \dfrac{\frac{4!}{\ln 3}}{3!}x^3 + \cdots + (-1)^n \dfrac{\frac{(n+1)!}{\ln n}}{n!}x^n + \cdots$

$= 1 - 2x + \dfrac{3}{\ln 2}x^2 - \dfrac{4}{\ln 3}x^3 + \cdots + (-1)^n \dfrac{n+1}{\ln n}x^n + \cdots$

② $\displaystyle\int_0^x \left[1 - 2t + \frac{3}{\ln 2}t^2 - \frac{4}{\ln 3}t^3 + \cdots + (-1)^n \frac{(n+1)}{\ln n}t^n + \cdots \right] dt$

$= \left[t - t^2 + \dfrac{t^3}{\ln 2} - \dfrac{t^4}{\ln 3} + \cdots + (-1)^n \dfrac{t^{n+1}}{\ln n} + \cdots \right]_0^x$

$= x - x^2 + \dfrac{x^3}{\ln 2} - \dfrac{x^4}{\ln 3} + \cdots + (-1)^n \dfrac{x^{n+1}}{\ln n} + \cdots$

③ $g(x)$ *is a convergent alternating series.*

$|R_n| < |a_{n+1}| < 0.002 \Rightarrow \dfrac{(0.3)^4}{\ln 3} = 0.007 \quad , \quad \dfrac{(0.3)^5}{\ln 4} = 0.0017$

$g(0.3) \approx (0.3) - (0.3)^2 + \dfrac{(0.3)^3}{\ln 2} - \dfrac{(0.3)^4}{\ln 3} = 0.242$

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